

$O(n)$ vector model at $n = -1, -2$ on random planar lattices: a direct combinatorial derivation

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The $O(n)$ vector model with logarithmic action on a lattice of coordination 3 is related to a gas of self-avoiding loops on the lattice. This formulation allows for analytical continuation in n : critical behaviour is found in the real interval $[-2, 2]$. The solution of the model on random planar lattices, recovered by random matrices, also involves an analytic continuation in the number n of auxiliary matrices. Here we show that, in the two cases $n = -1, -2$, a combinatorial reformulation of the loop gas problem allows to achieve the random matrix solution with no need of this analytical continuation.

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I. INTRODUCTION

Because of conformal symmetry, universality in 2-dimensional statistical systems on a regular lattice is a well-understood topic [1]. To perform a further average over the ensemble of random planar lattices, in some cases, provides even deeper results, because of the still larger symmetry (the discrete version of general covariance) [2, 3, 4].

The generating function of statistical configurations over random lattices can be written as the Feynman expansion of a proper action: the replacement of real or complex bosonic fields with $N \times N$ symmetric- or hermitean-matrix fields allows to count lattices of genus g with a weight proportional to N^{-g} , and a large- N limit, achieved via steepest descent or continuous approximation of matrix spectra, gives the restriction to planar lattices (see [5] for a recent pedagogical introduction).

Since the pioneering works on Ising and Potts Models [6], almost all interesting statistical 2-dimensional problems have been studied also in their variant on random planar lattices. The specific case of $O(n)$ vector model with logarithmic action on coordination-3 lattices, introduced by Nienhuis on the regular hexagonal lattice [7], has been solved in the random case by Kostov [8], and further studied in particular by [9, 10].

For a given lattice Λ , with V vertices, the $O(n)$ partition function is here defined as

$$Z_n(\Lambda, \beta) = \int \prod_{i=1}^V d^n \mathbf{s}_i \frac{2\delta(|\mathbf{s}_i|^2 - 1)}{\Omega_n} \prod_{\langle ij \rangle} (1 + \beta n \mathbf{s}_i \cdot \mathbf{s}_j) \quad (1)$$

where Ω_n is the surface of the unit sphere S^{n-1} in n -

dimensions, so that

$$\Omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}, \quad (2)$$

and it is introduced in order to have $Z_n(\Lambda, 0) = 1$. $O(n)$ -invariance implies

$$\langle 1 \rangle = 1; \quad \langle s_i^{(\alpha)} s_i^{(\beta)} \rangle = \frac{1}{n} \delta_{\alpha, \beta}; \quad (3)$$

$$\langle s_i^{(\alpha)} \rangle = 0; \quad \langle s_i^{(\alpha)} s_i^{(\beta)} s_i^{(\gamma)} \rangle = 0. \quad (4)$$

Note that higher-order momenta are not relevant, as they do not appear in the polynomial expansion of (1), this being a specific feature of the logarithmic action and the choice of the coordination number.

In each monomial of the expansion, each edge (i, j) can contribute either with a factor 1, or with a factor $\beta n s_i^{(\alpha)} s_j^{(\alpha)}$, with $\alpha \in \{1, \dots, n\}$. In this last case, we interpret an edge (i, j) as marked, and labeled by α .

In order to have a non-vanishing contribution after integration, each vertex must have an even number of marked edges of each species stemming from it, thus, as the coordination is 3, the only contributing configurations are the ones of self-avoiding unoriented loops, coloured with an index α . Summing over possible loop labelings leads to a factor n per loop, and we are left with a sum over configurations L of self-avoiding unlabeled unoriented loops

$$Z_n(\Lambda, \beta) = \sum_L n^{k(L)} \beta^{|L|}, \quad (5)$$

where $k(L)$ is the number of loops and $|L|$ is the number of edges in L . In this form of loop gas, the problem has a natural analytic extension to complex values of n . Note that we can alternatively sum over oriented loop configurations L^*

$$Z_n(\Lambda, \beta) = \sum_{L^*} \left(\frac{n}{2}\right)^{k(L^*)} \beta^{|L^*|}. \quad (6)$$

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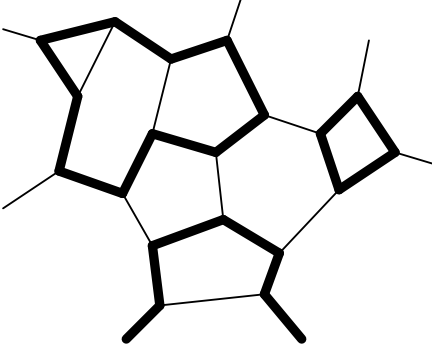


Figure 1: A portion of a typical configuration of loop gas on a random 3-graph.

A typical configuration over a random planar lattice looks like the one in figure 1. The model shows a nontrivial critical behaviour in the range $-2 \leq n \leq 2$, the cases $n = 1$ and $n = 2$ corresponding respectively to Ising and XY Model. In the case $n \rightarrow 0$ only one loop survives and when $\beta = \infty$ this model deals with Hamiltonian cycles [11]. The case $n = -1$ can also be seen as a particular $q \rightarrow 0$ limit of the q -state Potts Model [12]. Recently, it has also been connected with the generating function of forests on the graph [13].

In this letter we will show that, in the two cases $n = -1, -2$, a direct combinatorial reformulation of the loop gas problem allows to achieve the random matrix solution with no need of analytical continuation in n .

II. SKETCH OF THE RANDOM MATRIX SOLUTION

Consider the set \mathcal{L} of planar graphs Λ with coordination 3. We are interested in the double-series generating function

$$Z_n(g, \beta) = \sum_{\Lambda \in \mathcal{L}} \frac{g^{|\Lambda|}}{|\text{Aut}(\Lambda)|} Z_n(\Lambda, \beta), \quad (7)$$

where $|\Lambda|$ is the number of vertices in the lattice. When dealing with the $O(n)$ model, $n+1$ matrix fields M, E_1, \dots, E_n are introduced [8, 9, 10]. The quadratic part of the action is

$$\mathcal{S}_0 = \text{tr} \left[\frac{1}{2} (M^2 + E_1^2 + \dots + E_n^2) \right]; \quad (8)$$

while the interaction part is

$$\mathcal{S}_I = \text{tr} \left[-\frac{g}{3} M^3 - g\beta (E_1^2 + \dots + E_n^2) M \right]; \quad (9)$$

and the partition function (7) is given by

$$Z_n(g, \beta) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \ln \int \mathcal{D}_N(M, \{E_\alpha\}) e^{-N(\mathcal{S}_0 + \mathcal{S}_I)}. \quad (10)$$

Notice that auxiliary matrices E_α only appear quadratically, although with a non-trivial tensorial form in matrix indices. At this point one can apply a change of variables. Say $M = O\vec{\lambda}O^{-1}$, with $O \in O(N)$ and $\vec{\lambda} = \text{diag}(\lambda_1, \dots, \lambda_N)$. Change variables from E_α to $E'_\alpha = O^{-1}E_\alpha O$. Integration over M can be replaced by integration over $\vec{\lambda}$:

$$\int \mathcal{D}_N(M) = \int d^N \vec{\lambda} \Delta^2(\vec{\lambda}), \quad (11)$$

with the Jacobian given by the square Vandermonde determinant

$$\Delta^2(\vec{\lambda}) = \prod_{i \neq j} |\lambda_i - \lambda_j|. \quad (12)$$

We are left with

$$Z_n(g, \beta) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \ln \int d^N \vec{\lambda} \Delta^2(\vec{\lambda}) \int \mathcal{D}_N(\{E_\alpha\}) e^{-N(\mathcal{S}'_0 + \mathcal{S}'_I)}, \quad (13)$$

with

$$\mathcal{S}'_0 = \frac{1}{2} \sum_i \lambda_i^2 + \frac{1}{2} \sum_{i,j} \sum_{\alpha=1}^n (E_\alpha)_{ij} (E_\alpha)_{ji}; \quad (14)$$

$$\mathcal{S}'_I = -\frac{g}{3} \sum_i \lambda_i^3 + \beta g \sum_{i,j} \sum_{\alpha=1}^n (E_\alpha)_{ij} (E_\alpha)_{ji} \lambda_i. \quad (15)$$

We integrate over variables $(E_\alpha)_{ij}$, and, up to an irrelevant constant, we are left with

$$Z_n(g, \beta) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \ln \int d^N \vec{\lambda} \Delta^2(\vec{\lambda}) \exp \left[-N \left(\frac{1}{2} \sum_i \lambda_i^2 - \frac{g}{3} \sum_i \lambda_i^3 \right) \right] \prod_{i,j} (1 - \beta g (\lambda_i + \lambda_j))^{-\frac{n}{2}} \quad (16)$$

This result is, for example, the starting point for all the following analysis in Kostov article [8].

III. ALTERNATIVE COMBINATORICS FOR LOOP GAS

For a given lattice Λ of coordination 3, call \mathcal{P} (respectively \mathcal{P}^*) the set of configurations P of unoriented (respectively oriented) self-avoiding open paths, and $|P|$ the number of edges in P . Say that $P \subseteq L$ if all the edges in P are edges of the loop L , and $P^* \subseteq L^*$ if also orientation is preserved.

Introduce the two partition functions

$$Z(\Lambda, \beta, \gamma) = \sum_P \beta^{|P|} \sum_{L \supseteq P} \gamma^{|L| - |P|} \quad (17a)$$

$$Z^*(\Lambda, \beta, \gamma) = \sum_{P^*} \beta^{|P^*|} \sum_{L^* \supseteq P^*} \gamma^{|L^*| - |P^*|} \quad (17b)$$

Say L (resp. L^*) is composed by loops $\{\ell_k\}$, of $|\ell_k|$ edges, then the quantities (17) are equal respectively to

$$Z(\Lambda, \beta, \gamma) = \sum_L \prod_k \left((\gamma + \beta)^{|\ell_k|} - \beta^{|\ell_k|} \right) \quad (18a)$$

$$Z^*(\Lambda, \beta, \gamma) = \sum_{L^*} \prod_k \left(2(\gamma + \beta)^{|\ell_k|} - 2\beta^{|\ell_k|} \right) \quad (18b)$$

that is, each edge of L (resp. L^*) can be in P (resp. P^*) or not, independently, but for the global constraint that not all the edges of a given loop are in P (resp. P^*), which accounts for the subtractions in formulas (18). Remark that

$$Z(\Lambda, \beta, -\beta) = Z_{-1}(\Lambda, \beta) \quad (19)$$

$$Z^*(\Lambda, \beta, -\beta) = Z_{-2}(\Lambda, \beta) \quad (20)$$

IV. ALTERNATIVE RANDOM MATRIX FORMULATION OF THE PROBLEM

Now we want to restate the combinatorial theory described in section III, averaged over the set of random planar 3-graphs, as a random-matrix integral. We start from the case $n = -1$:

$$Z_{-1}(g, \beta) = \sum_{\Lambda \in \mathcal{L}} \frac{g^{|\Lambda|}}{|\text{Aut}(\Lambda)|} Z(\Lambda, \beta, -\beta). \quad (21)$$

Apparently, this theory requires three symmetric matrices: a matrix M for edges not in L , a matrix E for edges in L , but not in P , and a matrix F for edges in P . This is not the case: the constraint that F -edges do not make loops is not local, and cannot be implemented by a local action. It is necessary to build effective vertices of arbitrary degree, containing the connected components of P , in order to satisfy the constraint. So, as all F -edges are treated explicitly in vertex-combinatorics, they do not play a role in Feynman diagrammatic, and both the integration and the action only depend over matrices M and E , the quadratic part of the action being

$$\mathcal{S}_0 = \text{tr} \left[\frac{1}{2} (M^2 + E^2) \right]; \quad (22)$$

As in the previous treatment, we have the pure M - and E -edge vertices

$$\mathcal{S}_I^{(1)} = \text{tr} \left[-\frac{g}{3} M^3 + g\beta E^2 M \right], \quad (23)$$

where the factor $-\beta$ in the second term accounts for the two external E -legs. This corresponds to vertices of Λ which are not adjacent to edges of P . In the other case, say that a path of P has $p \geq 2$ vertices (in order to be non-empty): the corresponding vertex of the action must be built shrinking all the edges stemming from the path into a single point. The weight of the effective vertex is $-(\beta g)^p$. It has p external M -edges, and two E -edges,

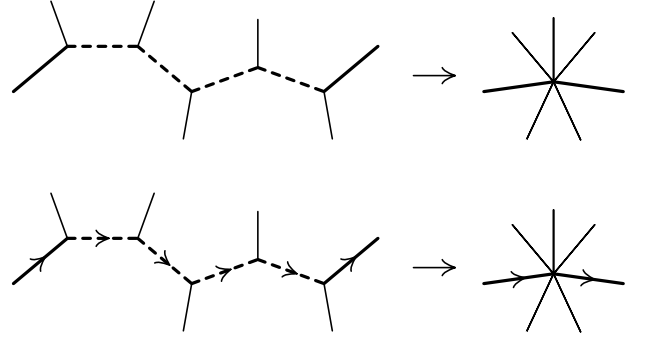


Figure 2: Contraction of the effective vertex, for the theory at $n = -1$ (top) and $n = -2$ (bottom). Thin, thick-solid and thick-dashed lines correspond respectively to M -, E - and F -edges.

in a certain cyclic sequence. Summing over all possible orderings gives a combinatorics

$$-\frac{(\beta g)^p}{2} \sum_{p'=0}^p \binom{p}{p'} EM^{p'} EM^{p-p'} \quad (24)$$

and thus a further term in the interaction part of the action is

$$\mathcal{S}_I^{(2)} = \text{tr} \sum_{p=2}^{\infty} \frac{(\beta g)^p}{2} \sum_{p'=0}^p \binom{p}{p'} EM^{p'} EM^{p-p'}. \quad (25)$$

Again the change of variables $M = O\vec{\lambda}O^{-1}$, $E' = O^{-1}EO$ allows to perform first an angular integration, then the gaussian integration of auxiliary variables E_{ij} . The various terms in the action combine to give

$$\begin{aligned} \mathcal{S}' &= \frac{1}{2} \sum_i \lambda_i^2 - \frac{g}{3} \sum_i \lambda_i^3 \\ &+ \frac{1}{2} \sum_{i,j} \sum_{p=0}^{\infty} (\beta g)^p \sum_{p'=0}^p \binom{p}{p'} E_{ij} \lambda_j^{p'} E_{ji} \lambda_i^{p-p'} \\ &= \frac{1}{2} \sum_i \lambda_i^2 - \frac{g}{3} \sum_i \lambda_i^3 + \frac{1}{2} \sum_{i,j} \frac{E_{ij} E_{ji}}{1 - \beta g(\lambda_i + \lambda_j)}, \end{aligned} \quad (26)$$

and a final Gaussian integration gives the quantity (16), for $n = -1$.

The case $n = -2$ is very similar. We have to calculate

$$Z_{-2}(g, \beta) = \sum_{\Lambda \in \mathcal{L}} \frac{g^{|\Lambda|}}{|\text{Aut}(\Lambda)|} Z^*(\Lambda, \beta, -\beta). \quad (27)$$

Again we have to work only with two matrices, M and E , but now, as edges in L are oriented, E is an hermitean-matrix field, the quadratic part of the action being

$$\mathcal{S}_0 = \text{tr} \left[\frac{1}{2} M^2 + \bar{E} E \right]; \quad (28)$$

while the pure M - and E -edge vertices give

$$\mathcal{S}_I^{(1)} = \text{tr} \left[-\frac{g}{3} M^3 + g\beta \bar{E} E M \right]. \quad (29)$$

For the remaining part of the action, say that an oriented path of P has $p \geq 2$ vertices. The weight of the effective vertex is still $-(\beta g)^p$, but now it has p external M -edges, one \bar{E} -edge and one E -edge, in a certain cyclic sequence. Summing over all possible orderings gives a slightly different combinatorics

$$-(\beta g)^p \sum_{p'=0}^p \binom{p}{p'} \bar{E} M^{p'} E M^{p-p'} \quad (30)$$

and thus a further term in the interaction part of the action is

$$\mathcal{S}_I^{(2)} = \text{tr} \sum_{p=2}^{\infty} (\beta g)^p \sum_{p'=0}^p \binom{p}{p'} \bar{E} M^{p'} E M^{p-p'}. \quad (31)$$

Again perform the change of variables, then angular integration: the various terms in the action combine to give

$$\begin{aligned} \mathcal{S}' &= \frac{1}{2} \sum_i \lambda_i^2 - \frac{g}{3} \sum_i \lambda_i^3 \\ &+ \sum_{i,j} \sum_{p=0}^{\infty} (\beta g)^p \sum_{p'=0}^p \binom{p}{p'} \bar{E}_{ij} \lambda_j^{p'} E_{ji} \lambda_i^{p-p'} \\ &= \frac{1}{2} \sum_i \lambda_i^2 - \frac{g}{3} \sum_i \lambda_i^3 + \sum_{i,j} \frac{\bar{E}_{ij} E_{ji}}{1 - \beta g(\lambda_i + \lambda_j)}, \end{aligned} \quad (32)$$

and a final Gaussian integration gives the quantity (16), for $n = -2$.

V. A FINAL REMARK

Calculations of section IV can be performed also for generic γ different from $-\beta$. We have

$$\begin{aligned} Z(g, \beta, \gamma) &= \lim_{N \rightarrow \infty} \frac{1}{N^2} \ln \int d^N \vec{\lambda} \Delta^2(\vec{\lambda}) \\ &\cdot \exp \left[-N \left(\frac{1}{2} \sum_i \lambda_i^2 - \frac{g}{3} \sum_i \lambda_i^3 \right) \right] \\ &\cdot \prod_{i,j} \left(\frac{1 - \beta g(\lambda_i + \lambda_j)}{1 - (\gamma + \beta)g(\lambda_i + \lambda_j)} \right)^{\frac{x}{2}}; \end{aligned} \quad (33)$$

with $x = 1$ for unoriented (resp. $x = 2$ for oriented) loops. More generally, if the weight associated to a loop of length ℓ is $\sum_B \beta_B^\ell - \sum_F \beta_F^\ell$, the whole procedure can be repeated, the last factor in the expression for the random lattice partition function being

$$\prod_{i,j} \left(\frac{\prod_F (1 - \beta_F g(\lambda_i + \lambda_j))}{\prod_B (1 - \beta_B g(\lambda_i + \lambda_j))} \right)^{\frac{x}{2}}. \quad (34)$$

We have denoted the positive weights β_B because we have seen how the bosonic matrices E 's naturally give rise to the jacobian in the denominator. Weights $-\beta_F$ could be obtained either introducing fermionic matrix fields, or by the combinatorial method discussed in this paper. Cancellations between numerator and denominator for equal values of β are a manifestation of Parisi-Sourlas supersymmetry [14].

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- [1] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory* (Springer-Verlag, New York, 1997).
 - [2] F. David, Nucl. Phys. B **257** [FS15] (1985) 45 and 543.
 - [3] J. Fröhlich, in: Lecture Notes in Physics, vol. 216, ed. L. Garrido (Springer, Berlin 1985); J. Ambjorn, B. Durhuus and J. Fröhlich, Nucl. Phys. B **257** [FS15] (1985) 433.
 - [4] V. A. Kazakov, I. K. Kostov and A. A. Migdal, Phys. Lett. B **157** (1985) 295; D. Boulatov, V. A. Kazakov, I. K. Kostov and A. A. Migdal, Phys. Lett. B **174** (1986) 87; Nucl. Phys. B **275** [FS17] (1986) 641.
 - [5] P. Di Francesco, Lectures given at the summer school “Applications of random matrices in physics”, Les Houches, June 2004, [arXiv:math-ph/0406013].
 - [6] V. A. Kazakov, Phys. Lett. A **119** (1986) 140 and Nucl. Phys. B **4** (Proc. Supp.) (1988) 93.
 - [7] B. Nienhuis, Phys. Rev. Lett. **49** (1982) 1062.
 - [8] I. Kostov, Mod. Phys. Lett. A **4** (1989) 217.
 - [9] M. Gaudin and I. Kostov, Phys. Lett. B **220** (1989) 200.
 - [10] B. Eynard and J. Zinn-Justin, Nucl. Phys. B **386** (1992) 558; [arXiv:hep-th/9204082]; B. Eynard and C. Kristjansen, Nucl. Phys. B **455** (1995) 577; [arXiv:hep-th/9506193]; B. Eynard and C. Kristjansen, Nucl. Phys. B **466** (1996) 463; [arXiv:hep-th/9512052]; B. Durhuus and C. Kristjansen, Nucl. Phys. B **483** (1997) 535; [arXiv:hep-th/9609008].
 - [11] B. Eynard, E. Guitter and C. Kristjansen, Nucl. Phys. B **528** (1998) 523; [arXiv:cond-mat/9801281]; E. Guitter, C. Kristjansen and J. L. Nielsen, Nucl. Phys. B **546** (1999) 731; [arXiv:cond-mat/9811289].
 - [12] J. L. Jacobsen, J. Salas, A. D. Sokal, [arXiv:cond-mat/0401026].
 - [13] S. Caracciolo, J. L. Jacobsen, H. Saleur, A. D. Sokal and A. Sportiello, Phys. Rev. Lett. **93** (2004) 080601 [arXiv:cond-mat/0403271].
 - [14] G. Parisi and N. Sourlas, Phys. Rev. Lett. **43** (1979) 744.